Extracting effects from non-linear models

Maarten L. Buis



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# Introduction

- What is the effect of $x$ on $y$?
  - Which effect do I choose: average marginal effects or marginal effects for someone with average values for the predictors or odds ratios, or...?
- Is the effect of $x$ on $y$ in group $a$ the same as the effect of $x$ on $y$ in group $b$?
  - How to interpret interaction effects: marginal effect for interaction effects or ratio of odds ratios?
  - Is such a comparison of effects across groups even identified?
- How much of the effect of $x$ on $y$ can I explain with variable $z$?
  - How can I get indirect/mediator effects?

# This introduction

- Quick review of
  - What is an effect?
  - What variables should we control for?
  - What is a non-linear model?

# What is an effect?

- Almost always a comparison of means.
- Say we have data on the income of a number of males and a number of *comparable* females.
- The comparison of the mean income of males and females gives us the effect of gender on income.
- This comparison can take the form of a difference: women earn on average $x$ euros/yen/pounds/dollars less than men,
- or it can take the form of a ratio: women earn on average $y$% less than men.

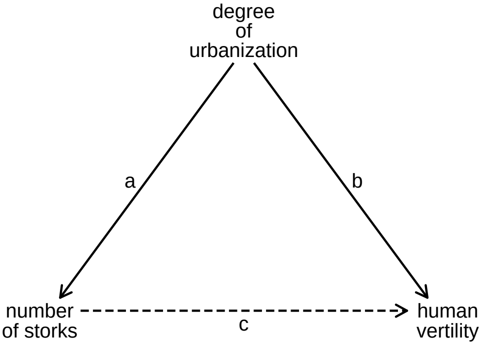
# OK, but what about continuous variables?

- Say we want to know the effect of age on income.
- Still a comparison of groups, each 1 year apart.
- Easiest solution is to constrain all these effects to be the same.
- The default for "difference effects" in linear regression.
- The default for "ratio effects" in non-linear regression with the log link-function.

# What variables do we need to control for?

- Confounding variables

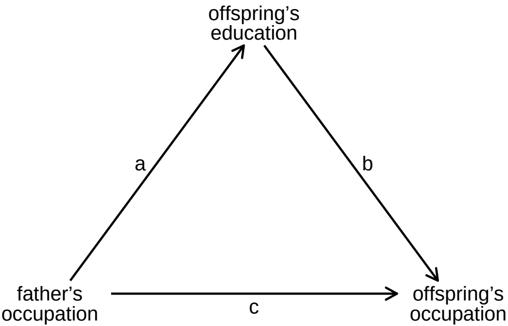
# What variables do we need to control for?



# What variables do we need to control for?

- Confounding variables
- Not intervening variables

# What variables do we need to control for?



## What variables do we need to control for?

- Confounding variables
- Not intervening variables
- Not idiosyncratic error/random noise/‘luck’
  - Many non-linear models exist to model a probability, an odds, a rate, or a hazard rate.
  - These concepts are defined by what we consider to be idiosyncratic error/random noise/‘luck’.
  - In these models the dependent variable is defined by what we choose not to control for.

# Non-linear models

- $f(\mathrm{E}(y)) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots$
- $f()$ is the link function, e.g.
    - logit link: $\log(\frac{u}{1-u})$
    - probit link: $\Phi(u)$
    - log link: $log(u)$
- an important characteristic of non-linear functions is that $f(\mathrm{E}[y]) \neq \mathrm{E}[f(y)]$
- Many non-linear models exist to accommodate
    - known bounds in the dependent variable, e.g. probability [0,1], odds, rate, hazard rate $\geq$ 0.
    - effects in terms of ratios

# Adjusted predictions (1)

- Say we want to know what the effect of having a college-degree on the probability of never being married, while controlling for age and whether or not the respondent lives in the South of USA.
- We do a logitstic regression:
```
sysuse nlsw88, clear
logit union collgrad age south
```
- An effect is a comparison of means, so why not get a predicted probability for a typical respondent with a college-degree and without a college-degree?

# Adjusted predictions (2)

- We could fix `age` and `south` at the mean and than predict the probability for the two groups:

# Adjusted predictions (2)

```
. margin , at(collgrad=(0 1) (mean) south age) noatlegend
Adjusted predictions                              Number of obs   =       2246
Model VCE    : OIM
Expression   : Pr(never_married), predict()
------------------------------------------------------------------------------
             |            Delta-method
             |     Margin   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         _at |
          1  |   .0874183   .0068627    12.74   0.000     .0739676     .100869
          2  |    .149139   .0154553     9.65   0.000     .1188472    .1794308
------------------------------------------------------------------------------
```

# Adjusted predictions (2)

- We could fix `age` and `south` at the mean and than predict the probability for the two groups:
- Alternatively, We could predict the probabilities for all individuals, and than compute the mean probabilities within each group:

# Adjusted predictions (2)

```
. margin , at(collgrad=(0 1)) noatlegend

Predictive margins                              Number of obs   =       2246
Model VCE    : OIM

Expression   : Pr(never_married), predict()

------------------------------------------------------------------------------
             |            Delta-method
             |     Margin   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         _at |
          1  |   .0893587   .0068779    12.99   0.000     .0758783     .102839
          2  |   .1517616   .0154767     9.81   0.000     .1214279    .1820954
------------------------------------------------------------------------------
```

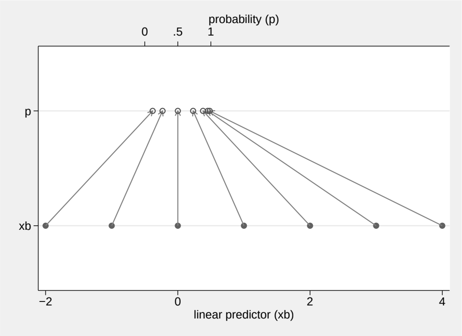
# Adjusted predictions (2)

- We could fix `age` and `south` at the mean and than predict the probability for the two groups:
- Alternatively, We could predict the probabilities for all individuals, and than compute the mean probabilities within each group:
- Why are these not the same?

# Adjusted predictions (3)

- The predicted probability is $\Lambda(xb)$, where
  - $\Lambda(u) = \frac{exp(u)}{1+exp(u)}$
  - $xb = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots$
- The first method consists of computing $\Lambda(\mathrm{E}[xb])$.
- The second method consists of computing $\mathrm{E}[\Lambda(xb)]$.

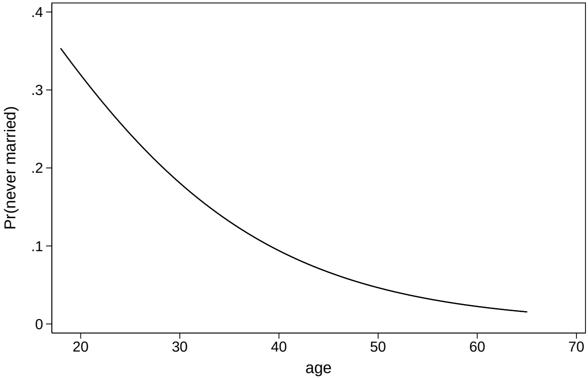
# Adjusted predictions (3)



# OK, but what about the effect of continuous variables?

- We want to summarize by how much the probability of being unmarried decreases when one gets a year older.
- This is a rate of change, or first derivative.
- In this context often called marginal effect.
- Problem: the relationship between age and the probability is non-linear, so there are many marginal effects

# Effect of age



# Effect of age (2)

```
. margins, dydx(age)  noatlegend                    ///
>          at((mean) collgrad south age=(35 45))

Conditional marginal effects                      Number of obs   =       2246
Model VCE    : OIM

Expression   : Pr(never_married), predict()
dy/dx w.r.t. : age

------------------------------------------------------------------------------
             |            Delta-method
             |      dy/dx   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
age          |
         _at |
          1  |  -.0086154   .0032458    -2.65   0.008     -.014977   -.0022538
          2  |  -.0046781   .0008927    -5.24   0.000    -.0064277   -.0029285
------------------------------------------------------------------------------
```

# OK, but what about the effect of continuous variables?

- We want to summarize by how much the probability of being unmarried decreases when one gets a year older.
- This is a rate of change, or first derivative.
- In this context often called marginal effect.
- Problem: the relationship between age and the probability is non-linear, so there are many marginal effects
- Problem: We get different effects when first fix the explanatory variables and than compute the marginal effect or first compute the marginal effects for each individual and than average.

# Effect of age (3)

```
. margins, dydx(age)  noatlegend                  ///
>          at((mean) collgrad south age=(35 45))

Conditional marginal effects                      Number of obs   =       2246
Model VCE    : OIM

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------------------------------------------------------------------------------
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------------------------------------------------------------------------------
```
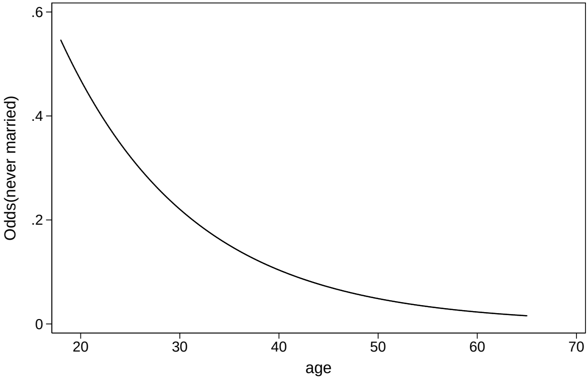
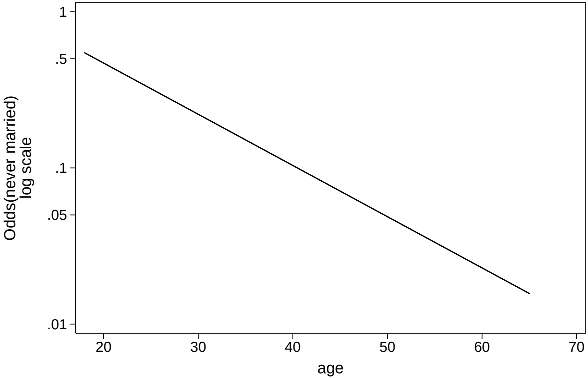

# Too many effects

- So, what is the "true" effect?
- In a strick sense none of them, but they are all valid approximations
- There is an alternative that is not an approximation when the link function contains a logarithm.
- In that case the effect in terms of ratios is assumed to be constant.

# Effect of age (4)



# Effect of age (5)



# Effect of age (3)

```
. gen byte baseline = 1

. gen c_age = age - 30

. logit never_married south c_age collgrad baseline, nolog nocons or

Logistic regression                               Number of obs   =       2246
                                                  Wald chi2(4)    =     940.91
Log likelihood = -736.65888                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
never_marr~d | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       south |   .8552155   .1219781    -1.10   0.273     .6466511    1.131048
       c_age |   .9272802   .0217551    -3.22   0.001     .8856065     .970915
    collgrad |   1.829794    .270653     4.08   0.000     1.369295    2.445159
    baseline |   .2041465   .0461335    -7.03   0.000     .1310948    .3179058
------------------------------------------------------------------------------
```

# Marginal effects of interaction effects

- An interaction between two variables is included by creating a new variable that is the product of the two.
- In linear regression we can interpret the multiplicative term as how much the effect of variable 1 changes for a unit change in variable 2 (and vice versa).
- Ai and Norton (2003) pointed out that this does not work for marginal effects in non-linear models.
- The aim is find out how much the effect of $x_1$ changes for a unit change in $x_2$
- i.e. the cross partial derivative with respect to $x_1$ and $x_2$.
- These can be computed in Stata by `inteff` and `inteff3`.

# Ratio effect of interaction effects

- The easier solution is to interpret interaction effects in terms of ratio effects.
- The interaction effect can now be interpreted as the ratio by which the effect of $x_2$ changes for a unit change in $x_1$

# Interaction effects

```
. sysuse nlsw88, clear
(NLSW, 1988 extract)

. gen byte high_occ = occupation < 3 if occupation < .
(9 missing values generated)

. gen byte black = race == 2 if race < .

. drop if race == 3
(26 observations deleted)

. gen byte baseline = 1

. logit high_occ black##collgrad baseline, or nocons nolog

Logistic regression                               Number of obs   =       2211
                                                  Wald chi2(4)    =     504.62
Log likelihood = -1199.4399                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
    high_occ | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     1.black |   .4194072   .0655069    -5.56   0.000     .3088072    .5696188
  1.collgrad |   2.465411    .293568     7.58   0.000     1.952238    3.113478
             |
      black# |
    collgrad |
        1 1  |   1.479715   .4132536     1.40   0.161     .8559637    2.558003
             |
    baseline |   .3220524   .0215596   -16.93   0.000     .2824512    .3672059
------------------------------------------------------------------------------
```

# Interaction effects

```
. margins , over(black collgrad) expression(exp(xb())) post

Predictive margins                              Number of obs   =       2211
Model VCE    : OIM

Expression   : exp(xb())
over         : black collgrad

------------------------------------------------------------------------------
             |            Delta-method
             |     Margin   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      black#|
    collgrad |
        0 0  |   .3220524   .0215596    14.94   0.000     .2797964    .3643084
        0 1  |   .7939914    .078188    10.15   0.000     .6407457    .9472371
        1 0  |   .1350711   .0190606     7.09   0.000      .097713    .1724292
        1 1  |   .4927536   .1032487     4.77   0.000       .29039    .6951173
------------------------------------------------------------------------------
```

# Interaction effects

```
. lincom 0.black#1.collgrad - 0.black#0.collgrad
 ( 1)  - 0bn.black#0bn.collgrad + 0bn.black#1.collgrad = 0
```

| | Coef. | Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| (1) | .471939 | .081106 | 5.82 | 0.000 | .3129742 | .6309038 |

```
. lincom 1.black#1.collgrad - 1.black#0.collgrad
 ( 1)  - 1.black#0bn.collgrad + 1.black#1.collgrad = 0
```

| | Coef. | Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| (1) | .3576825 | .1049933 | 3.41 | 0.001 | .1518994 | .5634656 |

# Latent variable interpretation of logistic regression

- Assume that there is some latent propensity of success $y^*$
- Someone gets a succes if $y^* > 0$ otherwise a failure.
- $y^* = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \varepsilon$
- the scale of $y^*$ is fixed by fixing the standard deviation of $\varepsilon$ to a fixed number $\frac{\pi}{\sqrt{3}}$.
- If we compare effects across groups or models we have to assume that the residual variance is equal otherwise the scale of the dependent variable will differ.

# Scenarios

- One way to get an idea about the size of this problem is to estimate various scenarios.
- The idea is that the heteroscedasticity comes from a (composite) unobserved variable, and to make assumptions regarding the size of the effect of this variable, its distribution, and how the effect changes when the observed variable of interest changes.
- The effect of the observed variables can than be estimated by integrating the likelihood function over this unobserved variable, which can be done by maximum simulated likelihood.
- This is implemented in Stata in `scenreg`.

# Probability and odds interpretation of logistic regression

- The problem is that the scale of $y^*$ is not defined
- We can solve that by interpreting the effects in terms of probabilities or odds, as these have a known scale.
- This does not do away with all arbitrariness:
  - the probability is defined in terms of what variables we chose to designate idiosyncratic error/luck
  - i.e. which variables we choose not to control for.
- Comparison of groups (interaction effects) can be solved this way, but comparisons of models with different explanatory variables (indirect effects) are still problematic.

# Problem with naïve method

```
. drop _all

. set obs 60000
obs was 0, now 60000

. gen z = ceil(_n / 20000) - 1

. bys z: gen x = ceil(_n  / 10000) - 1

. tab x z

           |                    z
         x |         0          1          2 |     Total
-----------+---------------------------------+----------
         0 |    10,000     10,000     10,000 |    30,000
         1 |    10,000     10,000     10,000 |    30,000
-----------+---------------------------------+----------
     Total |    20,000     20,000     20,000 |    60,000

. set seed 12345

. gen y = runiform() < invlogit(-4 + 4*x + 2*z)
```

# Problem with naïve method

```
. qui logit y x z
. est store direct
. local direct = _b[x]
.
. qui logit y x
. est store total
. local total _b[x]
.
. est tab direct total
-----------------------------------------
    Variable |    direct        total
-------------+---------------------------
           x |  4.0391332    2.6256242
           z |   2.026339
       _cons | -4.0452305   -1.3123133
-----------------------------------------

. di "naive indirect effect = " `total´ - `direct´
naive indirect effect = -1.413509
```

## `ldecomp` solution:

Say we want to find the indirect effect of college education through occupation on union membership.

- Estimate a logistic regression with all variables.
- Predict the log odds for each respondent and transform these to proportions.
- Compute the average proportion for college-graduates and non-college-graduates, and transform back to log odds: the difference between these is the total effect.
- Compute the average proportion for college graduates, assuming they have the distribution of occupation of the non-college-graduates.
- The only difference between the college graduates and the counterfactual group is the distribution of occupation, so this difference represents the indirect effect.
- The distribution of occupation remains constant when comparing the counterfactual group with the non-college graduates, so this difference represents the direct effect.

# Example

```
. sysuse nlsw88, clear
(NLSW, 1988 extract)

. gen byte high = occupation < 3 if occupation <.
(9 missing values generated)

. gen byte middle = occupation >= 3 & occupation < 7 if occupation < .
(9 missing values generated)

. ldecomp union south, direct(collgrad) indirect(high middle) at(south 0) or
(running _ldecomp on estimation sample)

Bootstrap replications (50)
----+--- 1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
..................................................    50

Bootstrap results                         Number of obs      =      1869
                                          Replications       =        50

------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
1/0          |
       total |   1.657501   .1867359     4.49   0.000     1.329096     2.06705
   indirect1 |   .8958344   .0491377    -2.01   0.045     .8045225    .9975101
     direct1 |   1.850231   .2249036     5.06   0.000     1.458004    2.347974
   indirect2 |   .8872166   .0493551    -2.15   0.031     .7955693    .9894213
     direct2 |   1.868203   .2311406     5.05   0.000     1.465921    2.380881
------------------------------------------------------------------------------

in equation i/j (comparing groups i and j)
let the fist subscript of Odds be the distribution of the the indirect variable
let the second subscript of Odds be the conditional probabilities
Method 1: Indirect effect = Odds_ij/Odds_jj
            Direct effect = Odds_ii/Odds_ij
Method 2: Indirect effect = Odds_ii/Odds_ji
            Direct effect = Odds_ji/Odds_jj

value labels
   0 not college grad
   1 college grad
```

# Conclusion

- There are two things that make non-linear models more difficult than non-linear models
  - The dependent variable is related to the independent variables via a non-linear function
  - The dependent variable is not directly observed, but a function of our model
- Often we can prevent the problem by using “ratio effects” instead of “difference effects”
- Sometimes we can bypass this problem by using a linear model as an approximation.
- Sometimes we will just have to use more complicated methods (`ldecomp`, `scenreg`, `inteff`, `inteff3`)