Likelihood of `betafit`

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This document shows how to get from the probability density presented in my presentation to log likelihood function in `betareg_lf.ado`. In my presentation I stated that the probability density of a Beta distribution in the alternative parameterizations is:

$$f(y|\mu, \phi) \propto y^{\mu\phi-1}(y-1)^{(1-\mu)\phi-1}$$

The $\propto$ means that I have ignored a constant. The constant is something that depends on μ and ϕ but not on y, in this case:

$$\frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)}$$

Whereby Γ is the gamma function, a function similar to the factorial (!) function, but the gamma function can also deal with non-integers.

So the complete probability density function of the Beta distribution in the alternative parameterizations is:

$$f(y|\mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y^{\mu\phi-1}(y-1)^{(1-\mu)\phi-1}$$

If you fill in the value of y for the first observation you get the likelihood function for the first observation(L_1). If you take the log, you get the log likelihood (ℓ_1):

$$
\begin{aligned}
L_i(\mu,\phi|y_i) &= \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y_i^{\mu\phi-1}(y_i-1)^{(1-\mu)\phi-1} \\
\ell_i(\mu,\phi|y_i) &= \ln\left(\frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y_i^{\mu\phi-1}(y_i-1)^{(1-\mu)\phi-1}\right) \\
&= \ln[\Gamma(\phi)] - \ln[\Gamma(\mu\phi)] - \ln[\Gamma((1-\mu)\phi)] + [\mu\phi-1]\ln(y_i) + [(1-\mu)\phi-1]\ln(y_i-1)
\end{aligned}
$$

The last step uses the following rules for dealing with logarithms:

$$
\begin{aligned}
\ln\left(\frac{a}{b}\right) &= \ln(a) - \ln(b) \\
\ln(ab) &= \ln(a) + \ln(b) \\
\ln\left(a^b\right) &= b\ln(a)
\end{aligned}
$$

Finally note that ϕ has to remain larger than zero, and μ has to remain between zero and one. To achieve that we let Stata maximize functions of ϕ and μ which can take any positive or negative number and transform them in the likelihood function to the correct metric. So we let Stata maximize $\ln(\phi)$ (which we call `‘ln_phi’`) and wherever we see ϕ in the likelihood we write `exp(‘ln_phi’)`. Similarly, `‘mu’` isn't μ but transformed μ, and wherever we see μ in the likelihood we write `invlogit(‘mu’)`. `lngamma` is a built in function in Stata that gives the natural logarithm of the gamma function. So the likelihood function in Stata code becomes:

```
qui replace ‘lnf’ = ///
lngamma(exp(‘ln_phi’)) -  ///
lngamma(invlogit(‘mu’)*exp(‘ln_phi’)) - ///
lngamma((1-invlogit(‘mu’))*exp(‘ln_phi’)) + ///
(invlogit(‘mu’)*exp(‘ln_phi’)-1)*ln($S_MLy) + ///
((1-invlogit(‘mu’))*exp(‘ln_phi’)-1)*ln(1-$S_MLy)
```